

# Geodesic study of a charged black hole

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The behavior of the timelike and null geodesics of charged E. Ayón-Beato and A. Garcia (ABG) black hole are investigated. For circular and radial geodesics, we investigate all the possible motions by plotting the effective potentials for different parameters. In conclusion, we have shown that there is no phenomenon of *superradiance* in this case.

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## I. INTRODUCTION

A black hole can be characterized by its mass  $M$ , charge  $Q$  and angular momentum  $J$ . A rotating charged black hole is represented by Kerr-Newman metric while uncharged one can be represented as Kerr black hole. Again, if a non-rotating black hole has charged then it is called Reissner-Nordström black hole. For uncharged case, the black hole which has only mass dependency is called Schwarzschild black hole.

Recently, E. Ayón-Beato and A. Garcia [1] (ABG) gave a solution of the Einstein field equations with nonlinear electrodynamics, known as ABG black hole.

The static spherical symmetric space-time is described by the metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where  $f(r) = 1 - \frac{2M}{r} + \frac{2M}{r} \tanh(\frac{Q^2}{2Mr})$  with  $M$  and  $Q$  corresponding to Mass, electro-magnetic charge of the black hole respectively.

This solution corresponds to a regular black hole with mass  $M$  and electro-magnetic charge  $Q$  and avoids thus the singularity problem. Also, the metric behaves asymptotically as the Reissner-Nordström (RN) black hole solution. It is clear that the singularity of the RN solution, at  $r = 0$ , has been omitted and now it simply becomes the origin of the spherical coordinates. Several [2, 3] authors studied different types of black holes (like charged brane-world black holes, dyadosphere of a charged black hole etc.). I. Radinschi [4] calculate the

energy distribution of a charged ABG black hole by using the energy-momentum complexes of Einstein and Møller.

In this paper, we will analyze the behaviour of the timelike and null geodesics of charged E. Ayón-Beato and A. Garcia (ABG) black hole. For circular and radial geodesics, we will discuss the possible motions by plotting the effective potentials for various parameters. We also check the phenomenon of superradiance for an incident massless scalar field for such a black hole.

## II. THE GEODESICS

Now, the equation for the geodesics in the metric (1) is given

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\sigma\lambda}^\mu \frac{dx^\sigma}{d\tau} \frac{dx^\lambda}{d\tau} = 0 \quad (2)$$

From eqn.(2) we have

$$\begin{aligned} \frac{1}{f(r)} \left( \frac{dr}{d\tau} \right)^2 &= \frac{E^2}{f(r)} - \frac{J^2}{r^2} - L \\ r^2 \left( \frac{d\phi}{d\tau} \right) &= J \\ \frac{dt}{d\tau} &= \frac{E}{f(r)} \end{aligned} \quad (3)$$

where we assume the  $\theta = \frac{\pi}{2}$  plane and constants  $E$  and  $J$  are the energy per unit mass and angular momentum, respectively about an axis perpendicular to the invariant plane  $\theta = \frac{\pi}{2}$ . Here, the affine parameter is  $\tau$  and the Lagrangian,  $L$  has values 0 and 1, respectively for massless and massive particles.

Now, for radial geodesic ( $J=0$ )

$$\dot{r}^2 \equiv \left( \frac{dr}{d\tau} \right)^2 = E^2 - Lf(r) \quad (4)$$

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Using the above equations we get

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right)\right)^2 \left[1 - \frac{L}{E^2} \left(1 - \frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right)\right)\right] \quad (5)$$

#### A. Photonlike particle motion(L=0)

For photonlike particle, we have

$$\left(\frac{dr}{dt}\right)^2 = \left[1 - \frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right)\right]^2 \quad (6)$$

i.e.

$$\pm t = \int \frac{dr}{\left(1 - \frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right)\right)} \quad (7)$$

Considering,  $1 - \tanh(z) \approx z$ , for a certain nbd. of  $z$  where  $\frac{Q^2}{2Mr} = z$ , we get

$$\pm t = \frac{Q^2}{2Mz} - \frac{Q}{2} \ln \left| \frac{Q + 2Mz}{Q - 2Mz} \right|$$

i.e.

$$\pm t = r - \frac{Q}{2} \ln \left| \frac{Qr + Q^2}{Qr - Q^2} \right| \quad (8)$$

The relation between time and distance for this particle is shown in the Fig. 1.

Now, we get from eqn.(4) as

$$\dot{r}^2 \equiv \left(\frac{dr}{d\tau}\right)^2 = E^2$$

that implies the  $\tau - r$  relationship as

$$\pm E\tau = r$$

The variation of proper time ( $\tau$ ) with respect to the radial distance ( $r$ ) is shown in Fig.2.

#### B. Massive particle motion (L=1)

For this case we can write,

$$\left(\frac{dr}{dt}\right)^2 = \frac{1}{E^2} \left(1 - \frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right)\right)^2 \left[E^2 - 1 + \frac{2M}{r} - \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right)\right]$$

Assuming  $\frac{Q^2}{2Mr} = z$  we get

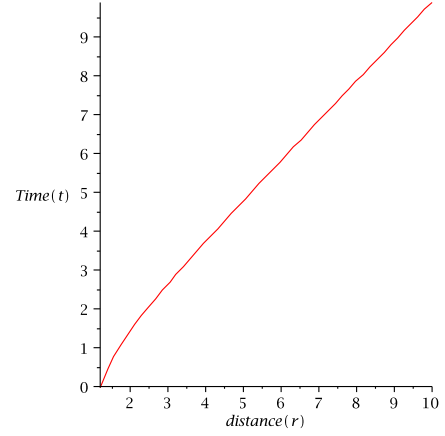


FIG. 1: Time-Distance relationship for photonlike particle (taking  $Q=1$ ).

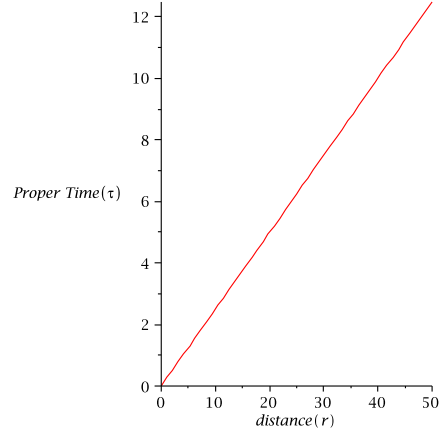


FIG. 2: Propertime-Distance relationship for photonlike particle (taking  $E=4$ ).

$$\pm t = -\frac{Q^2 E}{2M} \int \frac{dz}{z^2 \left(1 - \frac{4M^2}{Q^2} z(1 - \tanh z)\right) \sqrt{E^2 - 1 + \frac{4M^2}{Q^2} z(1 - \tanh z)}} \quad (9)$$

Again considering,  $1 - \tanh(z) \approx z$ , for a certain nbd. of  $z$  where  $\frac{Q^2}{2Mr} = z$ , we get

$$\pm t = -\frac{Q^2 E}{2M} \int \frac{dz}{z^2 \left(1 - \frac{4M^2}{Q^2} z^2\right) \sqrt{E^2 - 1 + \frac{4M^2}{Q^2} z^2}}$$

Again assuming,  $u = \frac{2M}{Q} z$  we get

$$\pm t = -\frac{QE}{\sqrt{E^2 - 1}} \int \frac{du}{u^2 (1 - u^2) \sqrt{1 + \frac{u^2}{E^2 - 1}}} \quad (10)$$

After simplification we get

$$\pm t = -QE \int \frac{dq}{q^2(1-E^2q^2)} \quad (11)$$

where  $q = \sin\left(\tan^{-1}\left(\frac{u}{\sqrt{E^2-1}}\right)\right)$

After integrating,

$$\pm t = Er\sqrt{E^2-1+\frac{Q^2}{2r^2}} - \frac{QE^3}{2} \ln \left| \frac{\sqrt{E^2-1+\frac{Q^2}{r^2}} + E\frac{Q}{r}}{\sqrt{E^2-1+\frac{Q^2}{r^2}} - E\frac{Q}{r}} \right| \quad (12)$$

This will give the time(t)-distance(r) relationship(Fig.3).

Again, from equation(4) we get

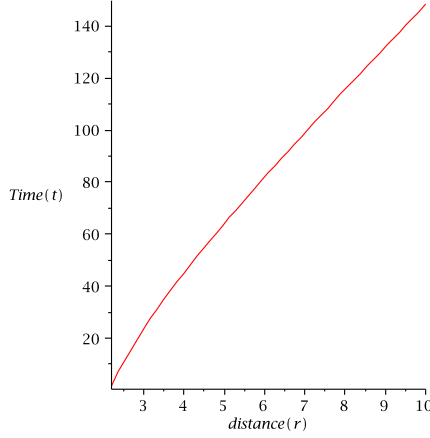


FIG. 3: Time-Distance relationship for massive particle(taking  $E=4, Q=1$ ).

$$\dot{r}^2 = \left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 - \frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right)\right) \quad (13)$$

After simplification in a similar manner we get

$$\pm \tau = \frac{r}{(E^2-1)Q^2} \sqrt{E^2-1+\frac{Q^2}{r^2}} \quad (14)$$

which gives the proper time( $\tau$ ) -distance(r) relationship(Fig.4).

### III. EFFECTIVE POTENTIAL

From the geodesic eqn.(2) and (3) we get

$$\frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2} \left[ E^2 - f(r) \left( \frac{J^2}{r^2} + L \right) \right] \quad (15)$$

If we compare eqn.(15) with  $\frac{\dot{r}^2}{2} + V_{eff} = 0$ , one can get the effective potential, which depends on E and L as follows :

$$V_{eff} = -\frac{1}{2} \left[ E^2 - f(r) \left( \frac{J^2}{r^2} + L \right) \right] \quad (16)$$

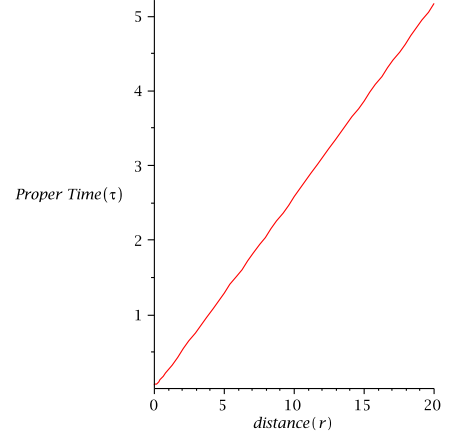


FIG. 4: Propertime-Distance relationship for massive particle(taking  $E=4, Q=1$ ).

#### A. For Photonlike particle( $L=0$ )

Consider, the radial geodesics where  $J = 0$ . Then, the corresponding  $V_{eff}$  is given by -

$$V_{eff} = -\frac{1}{2}E^2$$

The particle will behaves like a "free particle" i.e its  $V_{eff} = 0$  when energy,  $E = 0$ .

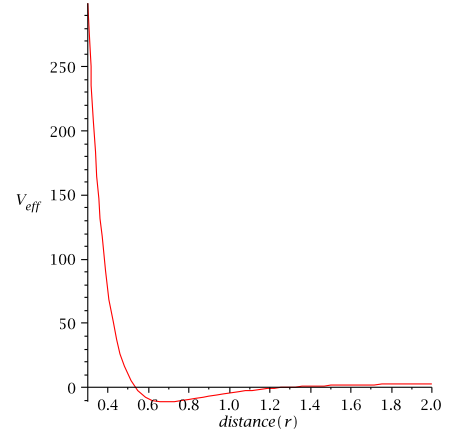


FIG. 5: Behaviour of the Effective Potential,  $V_{eff}$  for  $J \neq 0$  (taking  $E=1, J=10, Q=M=1$ )

We can easily say that the behavior of these geodesics is independent on the mass and charge of the black hole. Now, for circular geodesics,  $J \neq 0$ . The effective potential can be written as,

$$V_{eff} = -\frac{E^2}{2} + \frac{J^2}{2r^2} \left( 1 - \frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right) \right) \quad (17)$$

If  $r \rightarrow 0$ , the effective potential,  $V_{eff}(r)$  is large enough and approaches to  $-\frac{E^2}{2}$  when  $r \rightarrow \infty$ . At

the horizon, the effective potential,  $V_{eff} = -\frac{E^2}{2}$ . We assume, the effective potential for  $E = 0$  [in eqn.(17), put  $E = 0$ ]. The roots of the potential is nothing but the horizons for this case. It is nice to see the effective potential,  $V_{eff}$  is negative between the horizons. Hence, the particle would be bounded within the horizons. Again, a stable circular orbit will definitely exist between the horizons as  $V_{eff}$  has a minima there(Fig.5).

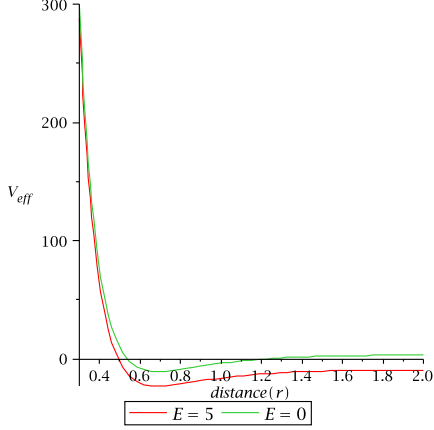


FIG. 6: Behaviour of the Effective Potential,  $V_{eff}$  (for radial geodesic) for massive particle(taking  $Q=M=1$ ).

#### B. For massive particle( $L=1$ )

The effective potential is given by

$$V_{eff} = -\frac{E^2}{2} + \frac{1}{2}f(r) \left( \frac{J^2}{r^2} + 1 \right) \quad (18)$$

where  $f(r) = 1 - \frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right)$ .

The roots of  $f(r)$  are the horizons. For radial geodesic( $J=0$ ),  $V_{eff}$  will vanish for some finite value of  $r$  in the region  $0 \leq r < r_-$  as  $f(r) > 0$ . Therefore, it is not possible for timelike geodesic to reach the singularity. The massive particle will emerge in other region avoiding the singularity and the spacetime is geodesically complete. We can studied the motion for both  $E = 0$  and  $E \neq 0$ . In first case, if we take  $E = 0$  then  $V_{eff}$  becomes

$$V_{eff} = \frac{1}{2} \left( 1 - \frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right) \right).$$

The roots of  $V_{eff}$  coincides with the horizons (Fig. 6). The shape of the potential indicates that the particle can move only inside the black hole. If we investigate the behaviour of  $V_{eff}$  for  $E \neq 0$ , then when  $r \rightarrow 0$  the effective potential becomes

$$V_{eff} \rightarrow -\frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right).$$

Now, for large  $r$ ,  $V_{eff} \rightarrow \frac{1-E^2}{2}$ . For a two horizons black hole, with in the two ranges  $0 \leq r < r_-$  and  $r_+ < r$ , the function  $f(r) > 0$ . Therefore, in these two region, it is possible for  $V_{eff}$  to have roots.

Now consider for non-zero angular momentum( $J \neq 0$ ) particle. Roots of  $V_{eff}$  coincides with the two horizons and the shape of  $V_{eff}$  is shown in Fig.6. Therefore, we can say that the massive particle with zero energy never escape from black hole and would be in a bound orbit. As the potential has a minima, the particle may have circular stable orbit.

Again, for large  $r$ ,  $V_{eff} \rightarrow \frac{1-E^2}{2}$  when  $E \neq 0$ . When  $r \rightarrow 0$  then the effective potential becomes

$$V_{eff} \rightarrow \frac{J^2}{2r^2} \left( 1 - \frac{2M}{r} + \frac{2M}{r} \tanh\left(\frac{Q^2}{2Mr}\right) \right).$$

The same conclusion can be taken for this case also that  $V_{eff}$  have finite roots in between  $0 \leq r < r_-$  and  $r_+ < r$ . Both the cases the massive particle would be in a bounded orbit.

#### IV. SOLUTION OF THE MASSLESS SCALAR WAVE EQUATION IN THE CHARGED E. AYÓN-BEATO AND GARCIA METRIC

In this section we shall analyze the scalar wave equation for charged E. Ayón-Beato and Garcia black hole geometry following Brill et al [5]. The wave equation for a massless particle is given by

$$g^{-1/2} \frac{\partial}{\partial x^\mu} \left( g^{1/2} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right) \chi = 0 \quad (19)$$

Here,  $g_{\mu\nu}$  is known from equation(1). Putting all the values one can obtain the following equation as

$$-\frac{r^4 \sin \theta}{\Delta} \frac{\partial^2 \chi}{\partial t^2} + \sin \theta \frac{\partial}{\partial r} \left( \Delta \frac{\partial \chi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \chi + \frac{1}{\sin \theta} \frac{\partial^2 \chi}{\partial \phi^2} = 0$$

Where  $\Delta = r^2 - 2Mr + 2Mr \tanh\left(\frac{Q^2}{2Mr}\right)$ .

Now, by using the separation of variable this equation can be solved as

$$\chi = e^{-i\omega t} e^{im\phi} R(r) \Theta(\theta)$$

Substituting this we get

$$\frac{r^4 \sin \theta}{\Delta} \omega^2 \chi + \frac{\sin \theta}{R} \frac{\partial}{\partial r} \left( \Delta \frac{\partial R}{\partial r} \right) \chi + \frac{1}{\Theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) \chi - \frac{m^2}{\sin \theta} \chi = 0$$

Now, the radial equation reduces to

$$\Delta \frac{\partial}{\partial r} \left( \Delta \frac{\partial R}{\partial r} \right) + (r^4 \omega^2 - \Delta \lambda) R = 0$$

and the angular part reduces to

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) - \frac{m^2}{\sin^2 \theta} \Theta + \lambda \Theta = 0$$

Now, if we substitute  $x = \cos \theta$ , the equation becomes

$$(1-x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d\Theta}{dx} - \left( \frac{m^2}{1-x^2} - \lambda \right) \Theta = 0 \quad (20)$$

If we now write  $\lambda = l(l+1)$  where  $l$  is an integer, then the equation

$$(1-x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d\Theta}{dx} + \left( l(l+1) - \frac{m^2}{1-x^2} \right) \Theta = 0$$

is the familiar associated Legendre equation and the solution is given by the associated Legendre polynomial  $P_l^m(x)$  and is expressed as

$$\Theta_l^m(\cos \theta) = P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$

### A. The Radial Equation: Absence of Superradiance

Kerr-Newman black hole allows energy extraction whereas Schwarzschild black hole prohibits. By an explicit process, this can be achieved (Penrose Process). Superradiance is nothing but the wave analogy of the Penrose process on black hole. When a bosonic or fermionic wave is incident on a black hole, naturally the reflected wave carries less energy than the incident one. But under specific condition, the transmitted wave, absorbed by the black hole carries negative energy into the black hole which makes the reflection coefficient greater than unity. This phenomena is called 'superradiance' [6]. According to this process the energy can be extracted from the black hole at the cost of losing its angular momentum. The required condition is

$$0 < \omega < m\omega_H \quad (21)$$

where  $\omega_H$  is the angular velocity of the horizon [7]. Basak et al discussed this phenomenon for the acoustic analogue of the Kerr black hole [8]. Shiraishi [9] and Ali [10] argued that superradiance phenomenon is possible if the black hole is rotating or charged. Now, we check whether the superradiance phenomenon really happens for E. Ayón-Beato and Garcia (ABG) black hole.

The radial equation can be written as

$$\Delta \frac{d}{dr} \left( \Delta \frac{dR}{dr} \right) + (\omega^2 r^4 - l(l+1)\Delta) R = 0$$

Now, we introduce the familiar tortoise coordinate,  $r^*$  defined as

$$\frac{dr^*}{dr} = \frac{r^2}{\Delta}$$

that implies

$$\Delta \frac{d}{dr} = r^2 \frac{d}{dr^*}$$

It is to be mentioned here that the variable  $r^*$  is constructed in the same way as in Schwarzschild or in Kerr metric. Therefore, in this case the variable is non-integrable. Though the basic purpose is still satisfied, the coordinate spans over the real line and pushes the horizon to  $-\infty$ .

Again, we introduce another function  $u(r) = rR$  which reduces the radial equation in a more familiar form

$$\frac{d^2 u}{dr^{*2}} + \left[ \frac{\Delta Q^2}{r^6} \text{sech}^2\left(\frac{Q^2}{2Mr}\right) + \frac{2\Delta M}{r^5} (1 - \tanh\left(\frac{Q^2}{2Mr}\right)) - \frac{2\Delta}{r^4} - \frac{l(l+1)\Delta}{r^4} + \frac{2\Delta^2}{r^6} + \omega^2 \right] u = 0$$

Therefore, a potential barrier remains where

$$V(r) = - \left[ \frac{\Delta Q^2}{r^6} \text{sech}^2\left(\frac{Q^2}{2Mr}\right) + \frac{2\Delta M}{r^5} (1 - \tanh\left(\frac{Q^2}{2Mr}\right)) - \frac{2\Delta}{r^4} - \frac{l(l+1)\Delta}{r^4} + \frac{2\Delta^2}{r^6} + \omega^2 \right]$$

At horizon ( $\Delta \rightarrow 0, r^* \rightarrow -\infty$ ), the radial equation comes out to be

$$\frac{d^2 u_H}{dr^{*2}} + \omega^2 u_H = 0$$

with  $V(r) = -\omega^2$ .

Now, asymptotically,  $r \rightarrow \infty$ , implies that  $r^* \rightarrow \infty$ . The equation has the same form as in the previous case

$$\frac{d^2 u_\infty}{dr^{*2}} + \omega^2 u_\infty = 0$$

Thus  $u_H = u_\infty$ , where  $u_H$  represents the radial solution at horizon and  $u_\infty$  is the solution at  $\infty$ . This equality shows that for a E. Ayón-Beato and Garcia (ABG) black hole there is *no phenomenon of superradiance for an incident massless scalar field*.

## V. CONCLUSION

In this investigation, we have analyzed the behavior of the timelike and null geodesics of charged E. Ayón-Beato and Garcia (ABG) Black hole. Here, we shown the behaviour of time-distance and proper time-distance graph. In case of radial geodesic, the effective potential is independent of the charge and mass of the black hole, for photonlike particle. However, from the shape of the potential, it is clear that the timelike particle can move only inside the black hole. On the other hand, for circular geodesics, the roots of the effective potential coincide with the horizon. Also, from Fig. 5, we can say that as the potential has a minima between the horizons, the

photonlike as well as timelike particles would be bounded in a stable circular orbit.

Though it is familiar that the superradiance phenomenon could be seen in charged or rotating black holes[9, 10], we have found that it is absent in the charged E. Ayón-Beato and Garcia (ABG) Black hole.

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